

Koszulity for graded skew PBW extensions

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Abstract

Pre-Koszul and Koszul algebras were defined by Priddy in [14]. There exist some relations between these algebras and the skew PBW extensions defined in [7]. In [23] we gave conditions to guarantee that skew PBW extensions over fields it turns out homogeneous pre-Koszul or Koszul algebra. In this paper we complement these results defining graded skew PBW extensions and showing that if R is a finite presented Koszul \mathbb{K} -algebra then every graded skew PBW extension of R is Koszul.

Key words and phrases. Graded skew PBW extensions, Koszul algebras, PBW algebras.

2010 *Mathematics Subject Classification.* 16S37, 16W50, 16W70, 16S36, 13N10.

1 Introduction

Pre-Koszul and Koszul algebras were introduced by Priddy in [14]. Koszul algebras have important applications in algebraic geometry, Lie theory, quantum groups, algebraic topology and combinatorics. The structure and history of Koszul algebras are detailed in [13]. There exist numerous equivalent definitions of a Koszul algebra (see for example [3]). Koszul algebras have been defined in a more general way by some authors (see for example [4], [5], [6], [11], [26]). Other authors have studied some properties of algebras constructed from Koszul algebras (see for example [8] and [20]). In this paper we will consider the classical notion of Koszulity introduced by Priddy.

Skew PBW extensions or σ -PBW extensions were defined in [7]. Several properties of these extensions have been recently studied (see for example [1], [10], [9], [15], [16], [17], [18], [24], [25]). There exist some relations between Koszul algebras with the skew PBW extensions of fields. In [23] we prove that every semi-commutative skew PBW extension of a field is Koszul. In the literature there exist examples of Koszul algebras which are skew PBW extensions of a \mathbb{K} -algebra $R \neq \mathbb{K}$. For example, the Jordan plane is an Artin-Schelter regular algebra of dimension two and therefore it is a Koszul algebra, but the Jordan plane is a PBW extension of $\mathbb{K}[x]$. Therefore, the results given in [23] does not apply in this case. We define graded skew PBW extensions and showed that every graded skew PBW extension of a finitely presented Koszul algebra is Koszul. Thus, our interest in this paper is to study the Koszul property (in the Priddy's sense) for graded skew PBW extensions. In the remainder of this paper, \mathbb{K} is a field and all algebras are \mathbb{K} -algebras.

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2 Graded skew PBW extensions

Let \mathbb{K} be a field. It is said that a \mathbb{K} -algebra A is *finitely graded* (see [19]) if the following conditions hold:

- (i) A is \mathbb{N} -graded (positively graded): $A = \bigoplus_{j \geq 0} A_j$,
- (ii) A is *connected*, i.e., $A_0 = \mathbb{K}$,
- (iii) A is *finitely generated* as \mathbb{K} -algebra, i.e., there is a finite set of elements $x_1, \dots, x_n \in A$ such that the set $\{x_{i_1}x_{i_2} \cdots x_{i_m} \mid 1 \leq i_j \leq n, m \geq 1\} \cup \{1\}$ spans A as a \mathbb{K} -space.

The algebra A is *augmented*, i.e., there is a canonical surjective \mathbb{K} -algebra homomorphism $\varepsilon : A \rightarrow \mathbb{K}$; $\ker(\varepsilon)$ is called *augmentation ideal*. A is called *locally finite* if $\dim_{\mathbb{K}} A_j < \infty$, for all $j \in \mathbb{N}$. A graded A -module $M = \bigoplus_{j \in \mathbb{Z}} M_j$ is called *locally finite* if $\dim_{\mathbb{K}} M_j < \infty$, for all $j \in \mathbb{Z}$. We say that the graded A -module M is *generated in degree s* if $M = A \cdot M_s$. M is *concentrated* in degree m if $M = M_m$. For any integer l , $M(l)$ is a graded A -module whose degree i component is $M(l)_i = M_{i+l}$.

The free associative algebra (tensor algebra) L in n generators x_1, \dots, x_n is the ring $L := \mathbb{K}\langle x_1, \dots, x_n \rangle$, whose underlying \mathbb{K} -vector space is the set of all words in the variables x_i , that is, expressions $x_{i_1}x_{i_2} \cdots x_{i_m}$ for some $m \geq 1$, where $1 \leq i_j \leq n$ for all j . The length of a word $x_{i_1}x_{i_2} \cdots x_{i_m}$ is m . We include among the words a symbol 1, which we think of as the empty word, and which has length 0. The product of two words is concatenation, and this operation is extended linearly to define an associative product on all elements. Note that L is positively graded with graduation given by $L := \bigoplus_{j \geq 0} L_j$, where $L_0 = \mathbb{K}$ and L_j spanned by all words of length j in the alphabet $\{x_1, \dots, x_n\}$, for $j > 0$; L is connected, the augmentation of L is given by the natural projection $\varepsilon : \mathbb{K}\langle x_1, \dots, x_n \rangle \rightarrow L_0 = \mathbb{K}$ and the augmentation ideal is given by $L_+ := \bigoplus_{j > 0} L_j$. Let P be a subspace of $F_2(L) := \mathbb{K} \oplus L_1 \oplus L_2$, the algebra $L/\langle P \rangle$ is called (nonhomogeneous) *quadratic algebra*. $L/\langle P \rangle$ is called *homogeneous quadratic algebra* if P is a subspace of L_2 , where $\langle P \rangle$ the two-sided ideal of L generated by P .

Proposition 2.1. *Let A be a connected \mathbb{N} -graded \mathbb{K} -algebra. A is finitely generated as \mathbb{K} -algebra if and only if $A = \mathbb{K}\langle x_1, \dots, x_m \rangle / I$, where I is a proper homogeneous two-sided ideal of $\mathbb{K}\langle x_1, \dots, x_m \rangle$. Moreover, for every $n \in \mathbb{N}$, $\dim_{\mathbb{K}} A_n < \infty$, i.e., A is locally finite.*

Proof. \Leftarrow): As the free algebra $L := \mathbb{K}\langle x_1, \dots, x_m \rangle$ is \mathbb{N} -graded and I is homogeneous, i.e., graded, then L/I is \mathbb{N} -graded with graduation given by $(L/I)_n := (L_n + I)/I$. Note that L/I is connected since $(L/I)_0 = \mathbb{K}$. Moreover, L/I is finitely generated as \mathbb{K} -algebra by the elements $x_i := x_i + I$, $1 \leq i \leq m$. Observe that L_n is finitely generated as \mathbb{K} -vector space, whence, $(L/I)_n$ is also finitely generated as \mathbb{K} -vector space, i.e., $\dim_{\mathbb{K}}((L/I)_n) < \infty$.

\Rightarrow): Let $a_1, \dots, a_m \in A$ be a finite collection of elements that generate A as \mathbb{K} -algebra; by the universal property of the free algebra $\mathbb{K}\langle x_1, \dots, x_m \rangle$, there exists a \mathbb{K} -algebra homomorphism $f : \mathbb{K}\langle x_1, \dots, x_m \rangle \rightarrow A$ with $f(x_i) := a_i$, $1 \leq i \leq m$; it is clear that f is surjective. Let $I := \ker(f)$, then I is a proper two-sided ideal of $\mathbb{K}\langle x_1, \dots, x_m \rangle$ and

$$A \cong \mathbb{K}\langle x_1, \dots, x_m \rangle / I. \quad (2.1)$$

Since A is \mathbb{N} -graded, we can assume that every a_i is homogeneous, $a_i \in A_{d_i}$ for some $d_i \geq 1$, moreover, at least one of generators is of degree 1. We define a new graduation for $L = \mathbb{K}\langle x_1, \dots, x_m \rangle$: we put weights d_i to the variables x_i and we set $L'_n := \mathbb{K}\langle x_{i_1} \cdots x_{i_m} \mid \sum_{j=1}^m d_{i_j} = n \rangle$ (the \mathbb{K} -space generated by $\{x_{i_1} \cdots x_{i_m} \mid \sum_{j=1}^m d_{i_j} = n\}$), $n \in \mathbb{N}$. This implies that f is graded, and from this we obtain that I is homogeneous. In fact, let $X_1 + \cdots + X_t \in I$, where $X_l \in L'_{n_l}$, $1 \leq l \leq t$, so $f(X_1) + \cdots + f(X_t) = 0$, and hence, $f(X_l) = 0$ for every l , i.e., $X_l \in I$. Finally, note that under the isomorphism \tilde{f} in (2.1), $\tilde{f}((L'_n + I)/I) = A_n$, so $\dim_{\mathbb{K}}(A_n) < \infty$. \square

Let A be a finitely graded algebra; it is said that A is *finitely presented* if the two-sided ideal I of relations in Proposition 2.1 is finitely generated.

We now recall the definition of skew PBW extension and some subclasses introduced in [7], [9] and [23]. We present also some key properties of these extensions.

Definition 2.2. Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a σ -PBW extension of R) if the following conditions hold:

- (i) $R \subseteq A$;
- (ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R -module, with basis the basic elements

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

- (iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \quad (2.2)$$

- (iv) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \quad (2.3)$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n \rangle$.

The notation $\sigma(R)\langle x_1, \dots, x_n \rangle$ and the name of the skew PBW extensions is due to the following proposition.

Proposition 2.3 ([7], Proposition 3). *Let A be a skew PBW extension of R . For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \rightarrow R$ and a σ_i -derivation $\delta_i : R \rightarrow R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \quad r \in R. \quad (2.4)$$

Remark 2.4. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension with endomorphisms σ_i , $1 \leq i \leq n$, as in the Proposition 2.3. We establish the following notation (see [7], Definition 6). $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$; $\sigma^\alpha := (\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n})$; $|\alpha| := \alpha_1 + \cdots + \alpha_n$; if $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$; for $X = x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. We have the following properties whose proof can be found in [7], Remark 2 and Theorem 7.

- (i) Each element $f \in A \setminus \{0\}$ has a unique representation as $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R \setminus \{0\}$ and $X_i \in \text{Mon}(A)$ for $1 \leq i \leq t$.
- (ii) For every $x^\alpha \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exists unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$.
- (iii) For every $x^\alpha, x^\beta \in \text{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$ where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Definition 2.5. Let A be a skew PBW extension of R , $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, where σ_i and δ_i ($1 \leq i \leq n$) are as in the Proposition 2.3

- (a) A is called *pre-commutative* if the conditions (iv) in Definition 2.2 are replaced by:
For any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R x_1 + \cdots + R x_n. \quad (2.5)$$

- (b) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.2 are replaced by

- (iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i; \quad (2.6)$$

- (iv') for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \quad (2.7)$$

- (c) A is called *bijective* if σ_i is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.
- (d) If $\sigma_i = \text{id}_R$ for every $\sigma_i \in \Sigma$, we say that A is a skew PBW extension of *derivation type*.
- (e) If $\delta_i = 0$ for every $\delta_i \in \Delta$, we say that A is a skew PBW extension of *endomorphism type*.
- (f) Any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$ will be called a *constant*. A is called *constant* if every element of R is constant.
- (g) A is called *semi-commutative* if A is quasi-commutative and constant.

Let $I \subseteq \sum_{n \geq 2} L_n$ be a finitely-generated homogeneous ideal of $\mathbb{K}\langle x_1, \dots, x_n \rangle$ and let $R = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, which is a connected-graded \mathbb{K} -algebra generated in degree 1. Suppose $\sigma : R \rightarrow R$ is a graded algebra automorphism and $\delta : R(-1) \rightarrow R$ is a graded σ -derivation (i.e. a degree +1 graded σ -derivation δ of R). Let $A := R[x; \sigma, \delta]$ be the associated *graded Ore extension* of R ; that is, $A = \bigoplus_{n \geq 0} R x^n$ as an R -module, and for $r \in R$, $xr = \sigma(r)x + \delta(r)$. We consider x to have degree 1 in A , and under this grading A is a connected graded algebra generated in degree 1 (see [6] and [12]). We introduce the definition of graded skew PBW extensions following [6].

Definition 2.6. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a bijective skew PBW extension of a \mathbb{N} -graded \mathbb{K} -algebra R . We said that A is a *graded skew PBW extension* if the following conditions hold:

- (i) x_1, \dots, x_n have degree 1 in A .
- (ii) σ_i is a graded ring homomorphism and $\delta_i : R(-1) \rightarrow R$ is a graded σ_i -derivation for all $1 \leq i \leq n$, where σ_i and δ_i are as in the Proposition 2.3.
- (iii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \dots + R_1 x_n$, as in (2.3) and $c_{i,j} \in R_0$.

Proposition 2.7. Let A be a graded skew PBW extension of R and let A_p the \mathbb{K} -space generated by the set

$$\left\{ r_t x^\alpha \mid t + |\alpha| = p, r_t \in R_t \text{ and } x^\alpha \in \text{Mon}(A) \right\},$$

for $p \geq 0$. Then:

- (i) $R_p \subseteq A_p$ for each $p \geq 0$.
- (ii) A is a graded \mathbb{K} -algebra with graduation

$$A = \bigoplus_{p \geq 0} A_p. \quad (2.8)$$

- (iii) A is a graded R -module with the above graduation.

Proof. Let $R = \bigoplus_{p \geq 0} R_p$ be a graded algebra and let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a graded skew PBW extension.

(i): If $r_p \in R_p$, then $r_p = r_p x_1^0 \cdots x_n^0 \in A_p$.
(ii): It is clear that $1 = x_1^0 \cdots x_n^0 \in A_0$. Let $f \in A \setminus \{0\}$, then by Remark 2.4, f has a unique representation as $f = r_1 X_1 + \dots + r_s X_s$, with $r_i \in R \setminus \{0\}$ and $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \in \text{Mon}(A)$ for $1 \leq i \leq s$. Let $r_i = r_{i_{q_1}} + \dots + r_{i_{q_m}}$ the unique representation of r_i in homogeneous elements of R . Then $f = (r_{1_{q_1}} + \dots + r_{1_{q_m}}) x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} + \dots + (r_{s_{q_1}} + \dots + r_{s_{q_u}}) x_1^{\alpha_{s1}} \cdots x_n^{\alpha_{sn}} = r_{1_{q_1}} x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} + \dots + r_{1_{q_m}} x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} + \dots + r_{s_{q_1}} x_1^{\alpha_{s1}} \cdots x_n^{\alpha_{sn}} + \dots + r_{s_{q_u}} x_1^{\alpha_{s1}} \cdots x_n^{\alpha_{sn}}$ is the unique representation of f in homogeneous elements of A . Therefore A is a direct sum of the family $\{A_p\}_{p \geq 0}$ of subspaces of A .

Now, let $x \in A_p A_q$. Without loss of generality we can assume that $x = (r_t x^\alpha)(r_s x^\beta)$ with $r_t \in R_t$, $r_s \in R_s$, $x^\alpha, x^\beta \in \text{Mon}(A)$, $t + |\alpha| = p$ and $s + |\beta| = q$. By Remark 2.4-(ii), we have that for r_s and x^α there exists unique elements $r_{s_\alpha} := \sigma^\alpha(r_s) \in R \setminus \{0\}$ and $p_{\alpha, r_s} \in A$ such that $x = r_t(r_{s_\alpha} x^\alpha + p_{\alpha, r_s}) x^\beta = r_t r_{s_\alpha} x^\alpha x^\beta + r_t p_{\alpha, r_s} x^\beta$, where $p_{\alpha, r_s} = 0$ or $\deg(p_{\alpha, r_s}) < |\alpha|$ if $p_{\alpha, r_s} \neq 0$. Now, by Remark 2.4-(iii), we have that for x^α , x^β there exists unique elements $c_{\alpha, \beta} \in R$ and $p_{\alpha, \beta} \in A$ such that $x = r_t r_{s_\alpha} (c_{\alpha, \beta} x^{\alpha+\beta} + p_{\alpha, \beta}) + r_t p_{\alpha, r_s} x^\beta = r_t r_{s_\alpha} c_{\alpha, \beta} x^{\alpha+\beta} + r_t r_{s_\alpha} p_{\alpha, \beta} + r_t p_{\alpha, r_s} x^\beta$, where $c_{\alpha, \beta}$ is left invertible, $p_{\alpha, \beta} = 0$ or $\deg(p_{\alpha, \beta}) < |\alpha| + |\beta|$ if $p_{\alpha, \beta} \neq 0$. We note that:

1. Since σ_i is graded for $1 \leq i \leq n$, then $\sigma_i^{\alpha_i}$ is graded and therefore σ^α is graded. Then $r_{s_\alpha} := \sigma^\alpha(r_s) \in R_s$ and $\delta_i^{\alpha_i}(r_s) \in R_{s+\alpha_i}$, for $1 \leq i \leq n$ and $\alpha_i \geq 0$.
2. $x_i^{\alpha_i} r_s = \sigma_i^{\alpha_i}(r_s) x_i^{\alpha_i} + \delta_i(\sigma_i^{\alpha_i-1}(r_s)) x_i^{\alpha_i-1} + \delta_i^2(\sigma_i^{\alpha_i-2}(r_s)) x_i^{\alpha_i-2} + \dots + \delta_i^j(\sigma_i^{\alpha_i-j}(r_s)) x_i^{\alpha_i-j} + \dots + \delta_i^{\alpha_i-1}(\sigma_i(r_s)) x_i + \delta_i^{\alpha_i}(r_s) \in A_{s+\alpha_i}$, since each summand in the above expression is in $A_{s+\alpha_i}$.

3. From the Definition 2.6, we have that for $1 \leq i < j \leq n$, $x_j x_i = c_{i,j} x_i x_j + r_{0_{ij}} + r_{1_{ij}} x_1 + \cdots + r_{n_{ij}} x_n \in A_2$. Then, for $1 \leq i < j < k \leq n$, we have that

$$\begin{aligned}
x_k(x_j x_i) &= x_k(c_{i,j} x_i x_j + r_{0_{ij}} + r_{1_{ij}} x_1 + \cdots + r_{n_{ij}} x_n) \\
&= (\sigma_k(c_{i,j}) x_k x_i x_j + \delta_k(c_{i,j}) x_i x_j) + (\sigma_k(r_{0_{ij}}) x_k + \delta_k(r_{0_{ij}})) \\
&\quad + (\sigma_k(r_{1_{ij}}) x_k x_1 + \delta_k(r_{1_{ij}}) x_1) + \cdots + (\sigma_k(r_{n_{ij}}) x_k x_n + \delta_k(r_{n_{ij}}) x_n) \\
&= \sigma_k(c_{i,j}) [c_{i,k} x_i x_k + r_{0_{ik}} + r_{1_{ik}} x_1 + \cdots + r_{n_{ik}} x_n] x_j + \delta_k(c_{i,j}) x_i x_j + \sigma_k(r_{0_{ij}}) x_k \\
&\quad + \delta_k(r_{0_{ij}}) + \sigma_k(r_{1_{ij}}) [c_{1,k} x_1 x_k + r_{0_{1k}} + r_{1_{1k}} x_1 + \cdots + r_{n_{1k}} x_n] \\
&\quad + \delta_k(r_{1_{ij}}) x_1 + \cdots + \sigma_k(r_{n_{ij}}) x_k x_n + \delta_k(r_{n_{ij}}) x_n \\
&= \sigma_k(c_{i,j}) c_{i,k} x_i [c_{j,k} x_j x_k + r_{0_{jk}} + r_{1_{jk}} x_1 + \cdots + r_{n_{jk}} x_n] + \sigma_k(c_{i,j}) r_{0_{ik}} x_j \\
&\quad + \sigma_k(c_{i,j}) r_{1_{ik}} x_1 x_j + \cdots + \sigma_k(c_{i,j}) r_{n_{ik}} x_n x_j + \delta_k(c_{i,j}) x_i x_j + \sigma_k(r_{0_{ij}}) x_k \\
&\quad + \delta_k(r_{0_{ij}}) + \sigma_k(r_{1_{ij}}) [c_{1,k} x_1 x_k + r_{0_{1k}} + r_{1_{1k}} x_1 + \cdots + r_{n_{1k}} x_n] \\
&\quad + \delta_k(r_{1_{ij}}) x_1 + \cdots + \sigma_k(r_{n_{ij}}) x_k x_n + \delta_k(r_{n_{ij}}) x_n \\
&= \sigma_k(c_{i,j}) c_{i,k} \sigma(c_{j,k}) x_i x_j x_k + \sigma_k(c_{i,j}) c_{i,k} \delta_i(c_{j,k}) x_j x_k + \sigma_k(c_{i,j}) c_{i,k} \sigma(c_{j,k}) \sigma_i(r_{0_{ij}}) x_i \\
&\quad + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{0_{ij}}) + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{1_{jk}}) x_i x_1 + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{1_{jk}}) x_1 + \cdots \\
&\quad + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{n_{jk}}) x_i x_n + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{n_{jk}}) x_n + \sigma_k(c_{i,j}) r_{0_{ik}} x_j \\
&\quad + \sigma_k(c_{i,j}) r_{1_{ik}} x_1 x_j + \cdots + \sigma_k(c_{i,j}) r_{n_{ik}} x_n x_j + \delta_k(c_{i,j}) x_i x_j + \sigma_k(r_{0_{ij}}) x_k + \delta_k(r_{0_{ij}}) \\
&\quad + \sigma_k(r_{1_{ij}}) c_{1,k} x_1 x_k + \sigma_k(r_{1_{ij}}) r_{0_{1k}} + \sigma_k(r_{1_{ij}}) r_{1_{1k}} x_1 + \cdots + \sigma_k(r_{1_{ij}}) r_{n_{1k}} x_n \\
&\quad + \delta_k(r_{1_{ij}}) x_1 + \cdots + \sigma_k(r_{n_{ij}}) x_k x_n + \delta_k(r_{n_{ij}}) x_n \\
&= \sigma_k(c_{i,j}) c_{i,k} \sigma(c_{j,k}) x_i x_j x_k + \sigma_k(c_{i,j}) c_{i,k} \delta_i(c_{j,k}) x_j x_k + \sigma_k(c_{i,j}) c_{i,k} \sigma(c_{j,k}) \sigma_i(r_{0_{ij}}) x_i \\
&\quad + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{0_{ij}}) + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{1_{jk}}) c_{1,i} x_1 x_i + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{1_{jk}}) r_{0_{1i}} \\
&\quad + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{1_{jk}}) r_{1_{1i}} x_1 + \cdots + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{1_{jk}}) r_{n_{1i}} x_n \\
&\quad + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{1_{jk}}) x_1 + \cdots + \sigma_k(c_{i,j}) c_{i,k} \sigma_i(r_{n_{jk}}) x_i x_n + \sigma_k(c_{i,j}) c_{i,k} \delta_i(r_{n_{jk}}) x_n \\
&\quad + \sigma_k(c_{i,j}) r_{0_{ik}} x_j + \sigma_k(c_{i,j}) r_{1_{ik}} x_1 x_j + \cdots + \sigma_k(c_{i,j}) r_{n_{ik}} c_{j,n} x_j x_n + \sigma_k(c_{i,j}) r_{n_{ik}} \\
&\quad + \sigma_k(c_{i,j}) r_{n_{ik}} r_{0_{jn}} + \sigma_k(c_{i,j}) r_{n_{ik}} r_{1_{jn}} x_1 + \cdots + \sigma_k(c_{i,j}) r_{n_{ik}} r_{n_{jn}} x_n + \delta_k(c_{i,j}) x_i x_j \\
&\quad + \sigma_k(r_{0_{ij}}) x_k + \delta_k(r_{0_{ij}}) + \sigma_k(r_{1_{ij}}) c_{1,k} x_1 x_k + \sigma_k(r_{1_{ij}}) r_{0_{1k}} + \sigma_k(r_{1_{ij}}) r_{1_{1k}} x_1 + \cdots \\
&\quad + \sigma_k(r_{1_{ij}}) r_{n_{1k}} x_n + \delta_k(r_{1_{ij}}) x_1 + \cdots + \sigma_k(r_{n_{ij}}) x_k x_n + \delta_k(r_{n_{ij}}) x_n.
\end{aligned}$$

Since all summands in the above sum have the form rx , where r is an homogeneous element of R , $x \in \text{Mon}(A)$ and $rx \in A_3$, we have that $x_k x_j x_i \in A_3$. Following this procedure we get in general that $x_{i_1} x_{i_2} \cdots x_{i_m} \in A_m$ for $1 \leq i_k \leq n$, $1 \leq k \leq m$, $m \geq 1$.

4. In a similar way and following the proof of [7], Theorem 7, we obtain that $x^\alpha r_s \in A_{|\alpha|+s}$, and since $c_{\alpha,\beta} \in R_0$, then $x^\alpha x^\beta \in A_{|\alpha|+|\beta|}$. Therefore $p_{\alpha,r_s} \in A_{|\alpha|+s}$ and $p_{\alpha,\beta} \in A_{|\alpha|+|\beta|}$. Then $r_t r_{s_\alpha} c_{\alpha,\beta} x^{\alpha+\beta} \in A_{t+s+|\alpha|+|\beta|}$, $r_t r_{s_\alpha} p_{\alpha,\beta} \in A_{t+s+|\alpha|+|\beta|}$ and $r_t p_{\alpha,r_s} x^\beta \in A_{t+|\alpha|+s+|\beta|}$, i.e., $x \in A_{p+q}$.

(iii): This follows from (ii). \square

Example 2.8. Quasi-commutative skew PBW extensions with the trivial graduation of R is a graded skew PBW extensions: Let $r \in R = R_0$, then $\sigma_i(r) = c_{i,r} \in R_0$, $\delta_i = 0$ and

$x_j x_i - c_{i,j} x_i x_j = 0 \in R_2 + R_1 x_1 + \cdots + R_1 x_n$; if we assume that R has a different graduation to the trivial graduation, then A is graded skew PBW extension provided that σ_i is graded and $c_{i,j} \in R_0$, $1 \leq i, j \leq n$.

Examples 2.9. Next we present specific examples of graded skew PBW extensions of the classical polynomial ring R with coefficients in a field \mathbb{K} , which are not quasi-commutative and where R has the usual graduation. In [7], [10] and [23] we can be found further details of these algebras.

1. The Jordan plane. $A = \mathbb{K}\langle x, y \rangle / \langle yx - xy - x^2 \rangle \cong \sigma(\mathbb{K}[x])\langle y \rangle$.
2. The homogenized enveloping algebra. $\mathcal{A}(\mathcal{G}) \cong \sigma(\mathbb{K}[z])\langle x_1, \dots, x_n \rangle$.
3. The Diffusion algebra 2. $A \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle D_1, \dots, D_n \rangle$.
4. The algebra $U \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle y_1, \dots, y_n; z_1, \dots, z_n \rangle$.
5. Manin algebra. $\mathcal{O}(M_q(2)) \cong \sigma(\mathbb{K}[u])\langle x, y, v \rangle$.
6. Algebra of quantum matrices. $\mathcal{O}_q(M_n(\mathbb{K})) \cong \sigma(\mathbb{K}[x_{im}, x_{jk}])\langle x_{ik}, x_{jm} \rangle$, for $1 \leq i < j, k < m \leq n$.
7. Quadratic algebras. If $a_1 = a_4 = 0$ then the quadratic algebra is a graded skew PBW extension of $R = \mathbb{K}[y, z]$, and if $a_5 = a_3 = 0$ then quadratic algebras are graded skew PBW extensions of $R = \mathbb{K}[x, z]$.

Remark 2.10. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a graded skew PBW extension. Then we immediately have the following properties:

- (i) A is a \mathbb{N} -graded \mathbb{K} -algebra and $A_0 = R_0$.
- (ii) R is connected if and only if A is connected.
- (iii) If R is finitely generated then A is finitely generated. Indeed, as $\text{Mon}(A) = \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is R -base for A , and R is finitely generated as \mathbb{K} -algebra, then there is a finite set of elements $t_1, \dots, t_s \in R$ such that the set $\{t_{i_1} t_{i_2} \cdots t_{i_m} \mid 1 \leq i_j \leq s, m \geq 1\} \cup \{1\}$ spans R as a \mathbb{K} -space. Then there is a finite set of elements $t_1, \dots, t_s, x_1, \dots, x_n \in A$ such that the set $\{t_{i_1} t_{i_2} \cdots t_{i_m} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 1 \leq i_j \leq s, m \geq 1, \alpha_1, \dots, \alpha_n \in \mathbb{N}\}$ spans A as a \mathbb{K} -space. So, if R is generated in degree 1 then A is generated in degree 1.
- (iv) For (i), (ii) and (iii) above we have that if R is a finitely graded algebra then A is a finitely graded algebra.
- (v) If R is locally finite, A as \mathbb{K} -algebra is a locally finite. Indeed, $\dim_{\mathbb{K}} A_0 = \dim_{\mathbb{K}} R_0$, $\dim_{\mathbb{K}} A_1 = \dim_{\mathbb{K}} R_1 + n$; let \mathcal{B}_t be a (finite) base of R_t , $t \geq 0$, then for a fixed $p \geq 2$ the set $\{r_t x^\alpha \mid t + |\alpha| = p, r_t \in \mathcal{B}_t \text{ and } x^\alpha \in \text{Mon}(A)\}$ is a finite base for A_p .
- (vi) A as R -module is locally finite.

- (vii) If A is quasi-commutative and R is concentrate in degree 0, then $A_0 = R$.
- (viii) If R is a homogeneous quadratic algebra then A is a homogeneous quadratic algebra.
- (ix) If R is finitely presented then A is finitely presented. Indeed: by Proposition 2.1, $R = \mathbb{K}\langle t_1, \dots, t_m \rangle / I$ where

$$I = \langle r_1, \dots, r_s \rangle \quad (2.9)$$

is a two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m \rangle$ generated by a finite set r_1, \dots, r_s of homogeneous polynomials in $\mathbb{K}\langle t_1, \dots, t_m \rangle$. Then $A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle / J$ where

$$J = \langle r_1, \dots, r_s, f_{hk}, g_{ji} \mid 1 \leq i, j, h \leq n, 1 \leq k \leq m \rangle \quad (2.10)$$

is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle$ generated by a finite set of homogeneous elements $r_1, \dots, r_s, f_{hk}, g_{ji}$ where r_1, \dots, r_s are as in (2.9);

$$f_{hk} := x_h t_k - \sigma_h(t_k) x_h - \delta_h(t_k) \quad (2.11)$$

with σ_h and δ_h as in Proposition 2.3;

$$g_{ji} := x_j x_i - c_{i,j} x_i x_j - (r_{0,j,i} + r_{1,j,i} x_1 + \dots + r_{n,j,i} x_n) \quad (2.12)$$

as in (2.3) of Definition 2.2.

Remark 2.11. The class of graded iterated Ore extensions \subsetneq class of graded skew PBW extensions. For example, the homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ and the Diffusion algebra 2 are graded skew PBW extension but this is not iterated Ore extensions. Therefore, the definition of graded skew PBW extensions is more general that the definition of graded Ore extensions.

3 Koszul algebras

Let $A = \mathbb{K} \oplus A_1 \oplus A_2 \oplus \dots$ be a locally finite graded algebra and $E(A) = \bigoplus_{s,p} E^{s,p}(B) = \bigoplus_{s,p} Ext_A^{s,p}(\mathbb{K}, \mathbb{K})$ the associated bigraded Yoneda algebra, where s is the cohomology degree and $-p$ is the internal degree inherited from the grading on A . Let $E^s(A) = \bigoplus_p E^{s,p}(A)$. A is called *Koszul* if the following equivalent conditions hold (see [13], Chapter 2, Definition 1):

- (i) $Ext_A^{s,p}(\mathbb{K}, \mathbb{K}) = 0$ for $s \neq p$;
- (ii) A is one-generated and the algebra $Ext_A^*(\mathbb{K}, \mathbb{K})$ it is generated by $Ext_A^1(\mathbb{K}, \mathbb{K})$, i.e., $E(A)$ is generated in the first cohomological degree;
- (iii) The module \mathbb{K} admits a *linear free resolution*, i.e., a resolution by free A -modules

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{K} \rightarrow 0$$

such that P_i is generated in degree i .

Let A be a graded Ore extension of R . Then A is homogeneous quadratic if and only if R is homogeneous quadratic. Furthermore, A is Koszul if and only if R is Koszul (see [12], Corollary 1.3).

Proposition 3.1. *The graded iterated Ore extension $A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is Koszul if and only if R is Koszul.*

Proof. Suppose

$$\sigma_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \rightarrow R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$$

is a graded algebra automorphism and

$$\delta_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}](-1) \rightarrow R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$$

is a graded σ_i -derivation, $2 \leq i \leq n$. Let $A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be the *graded iterated Ore extension* of R , where x_1, \dots, x_n have degree 1 in A . Then from [12], Corollary 1.3 the result is clear. \square

Proposition 3.2 ([10], Theorem 2.3). *Let A be a quasi-commutative skew PBW extension of a ring R . Then (i) A is isomorphic to an iterated skew polynomial ring, and (ii) if A is bijective, each endomorphism of the skew polynomial ring in (i) is an isomorphism.*

Proposition 3.3. *Let A be a graded quasi-commutative skew PBW extension of R . Then R is a Koszul algebra if and only if A is Koszul.*

Proof. If A is a graded quasi-commutative bijective skew PBW extension of R , then by Proposition 3.2 A is isomorphic to an iterated graded Ore extension wherein each endomorphism is bijective. Then by Proposition 3.1, R is Koszul if and only if A is Koszul. \square

4 PBW algebras

Let $L = \mathbb{K}\langle x_1, \dots, x_n \rangle$ and let $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle P \rangle$ be a homogeneous quadratic algebra with a fixed generators $\{x_1, \dots, x_n\}$. For a multindex $\alpha := (i_1, \dots, i_m)$, where $1 \leq i_k \leq n$, we denote the monomials $x^\alpha := x_{i_1}x_{i_2} \cdots x_{i_m} \in \mathbb{K}\langle x_1, \dots, x_n \rangle$. For $\alpha = \emptyset$ we set $x^\emptyset := 1$. Now let us equip the subspace L_2 with the basis consisting of the monomials $x_{i_1}x_{i_2}$. Let $S^{(1)} := \{1, 2, \dots, n\}$, $S^{(1)} \times S^{(1)}$ the cartesian product, then for $P \subseteq L_2$ we obtain the set $S \subseteq S^{(1)} \times S^{(1)}$ of pairs of indices (l, m) for which the class of x_lx_m in L_2/P is not in the span of the classes of x_rx_s with $(r, s) < (l, m)$, where $<$ denotes the lexicographical order. Hence, the relations in A can be written in the following form (see [13], Lemma 4.1.1):

$$x_ix_j = \sum_{\substack{(r,s) < (i,j) \\ (r,s) \in S}} c_{ij}^{rs} x_r x_s, \quad (i, j) \in S^{(1)} \times S^{(1)} \setminus S.$$

Define further $S^{(0)} := \{\emptyset\}$, and for $m \geq 2$,

$$S^{(m)} := \{(i_1, \dots, i_m) \mid (i_k, i_{k+1}) \in S, k = 1, \dots, m-1\}$$

and consider the monomials $\{x_{i_1} \cdots x_{i_m} \in A_m \mid (i_1, \dots, i_m) \in S^{(m)}\}$. Note that these monomials always span A_m as a vector space and the monomials

$$(A, S) := \{x_{i_1} \cdots x_{i_m} \mid (i_1, \dots, i_m) \in \cup_{m>0} S^{(m)}\} \quad (4.1)$$

linearly span the entire A . We call (A, S) in (4.1) a *PBW-basis* of A if they are linearly independent and hence form a \mathbb{K} -linear basis. The elements x_1, \dots, x_n are called *PBW-generators* of A . A *PBW-algebra* is a homogeneous quadratic algebra admitting a PBW-basis, i.e., there exists a permutation of x_1, \dots, x_n such that the standard monomials in x_1, \dots, x_n conform a \mathbb{K} -basis of A . In [23] we show that every semi-commutative skew PBW extension of \mathbb{K} is a PBW algebra.

Proposition 4.1. *Let A be a graded skew PBW extension of a finitely presented algebra R . If R is a PBW algebra then A is a PBW algebra.*

Proof. Let R be a finitely presented PBW algebra with PBW generators t_1, \dots, t_m . Then by Proposition 2.1, $R = L^t/I$, where $L^t = \mathbb{K}\langle t_1, \dots, t_m \rangle$ and

$$I = \langle r_1, \dots, r_s \rangle \quad (4.2)$$

is a two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m \rangle$ generated by a finite set r_1, \dots, r_s of homogeneous polynomials in $\mathbb{K}\langle t_1, \dots, t_m \rangle$ of degree two. Let

$$(R, S_t) := \{t_{i_1} \cdots t_{i_p} \mid (i_1, \dots, i_p) \in \cup_{p>0} S_t^{(p)}\} \quad (4.3)$$

be a PBW basis of R , with $S_t^{(p)} = \{(i_1, i_2, \dots, i_p) \mid (i_k, i_{k+1}) \in S_t, k = 1, \dots, p-1\}$, $S_t^{(1)} := \{1, 2, \dots, m\}$ and $S_t \subseteq S_t^{(1)} \times S_t^{(1)}$ is the set of pairs of indices (i_μ, i_ν) for which the class of $t_{i_\mu} t_{i_\nu}$ in L_2^t/P (where P is the space of relations r_1, \dots, r_s) is not in the span of the classes of $t_r t_s$ with $(r, s) < (i_\mu, i_\nu)$. For $1 \leq d \leq s$,

$$r_d = t_{i_d} t_{j_d} = \sum_{\substack{(r_d, q_d) < (i_d, j_d) \\ (r_d, q_d) \in S_t}} c_{i_d j_d}^{r_d q_d} t_{r_d} t_{q_d}, \quad (i_d, j_d) \in S_t^{(1)} \times S_t^{(1)} \setminus S_t. \quad (4.4)$$

Let $A = \sigma(R)\langle x_{m+1}, \dots, x_{m+n} \rangle$ be a graded skew PBW extension of R . As $R \subseteq A$, we have that $A = \mathbb{K}\langle t_1, \dots, t_m, x_{m+1}, \dots, x_{m+n} \rangle/J$ where

$$J = \langle r_1, \dots, r_s, f_{hk}, g_{ji} \mid m+1 \leq i, j, h \leq m+n, 1 \leq k \leq m \rangle \quad (4.5)$$

is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_{m+1}, \dots, x_{m+n} \rangle$ generated by a set $r_1, \dots, r_s, f_{hk}, g_{ji}$ where r_1, \dots, r_s are as in (4.2); let

$$f_{hk} := x_{m+h} t_k - \sigma_{m+h}(t_k) x_{m+h} - \delta_{m+h}(t_k) \quad (4.6)$$

with σ_{m+h} and δ_{m+h} as in Proposition 2.3;

$$g_{ji} := x_{m+j} x_{m+i} - c_{i,j} x_{m+i} x_{m+j} - (r_{0,j,i} + r_{1,j,i} x_{m+1} + \cdots + r_{n_j,i} x_{m+n}) \quad (4.7)$$

is as in (2.3) of Definition 2.2. As A is graded skew PBW extension then it is homogeneous quadratic, since $r_1, \dots, r_s, f_{hk}, g_{ji}$ are homogeneous polynomials of degree two in $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle$. Now, let $S_{tx}^{(1)} := \{1, \dots, m, m+1, \dots, m+n\}$. From the relations (4.6) we obtain the set $S_{tx} := \{(k, l) \mid 1 \leq k \leq m, m+1 \leq l \leq m+n\}$. From the relations (4.7) we obtain the set $S_x := \{(m+i, m+j) \mid 1 \leq i \leq j \leq n\}$. From Definition 2.2, we have that $R \subseteq A$ and A is a left free R -module. Then, for the \mathbb{K} -algebra A , we have that

$$S^{(p)} = \{(i_1, \dots, i_k, i_{k+1}, \dots, i_p) \mid (i_1, \dots, i_k) \in S_t^{(k)} \text{ and } i_{k+1} \leq \dots \leq i_p\}.$$

So,

$$(A, S) := \{t_{i_1} \cdots t_{i_k} x_{i_{k+1}} \cdots x_{i_p} \mid (i_1, \dots, i_k, i_{k+1}, \dots, i_p) \in \cup_{p>0} S^{(p)}\} \quad (4.8)$$

span A as a vector space. As $(R, S_t) := \{t_{i_1} \cdots t_{i_\nu} \mid (i_1, \dots, i_\nu) \in \cup_{p>0} S_t^{(p)}\}$ is a \mathbb{K} -basis for R and A is a left free R -module, with basis the basic elements

$$\begin{aligned} \{x^\alpha = x_{m+1}^{\alpha_{m+1}} \cdots x_{m+n}^{\alpha_{m+n}} \mid \alpha = (\alpha_{m+1}, \dots, \alpha_{m+n}) \in \mathbb{N}^n\} \\ = \{x_{i_{k+1}} \cdots x_{i_p} \mid m+1 \leq i_{k+1} \leq \dots \leq i_p \leq m+n\} \cup \{1\}, \end{aligned}$$

then (A, S) is a PBW basis of A . Therefore A is a PBW algebra. □

Remark 4.2. If in the free algebra $\mathbb{K}\langle x_1, \dots, x_n \rangle$ we fix the set $\{1, 2, \dots, n\}$, we implicitly understand that $x_1 < x_2 < \dots < x_n$. For example, for $A = \mathbb{K}\langle x, y, z \rangle / \langle z^2 - xy - yx, zx - xz, zy - yz \rangle$ with $x < y < z$, i.e., $x = x_1, y = x_2, z = x_3$, we have that $S^{(1)} = \{1, 2, 3\}$, $S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} = S^{(2)}$. Note that (A, S) is not a \mathbb{K} -basis for A . Indeed: $(2, 1, 1), (1, 1, 2) \in S^{(3)}$ and therefore the classes (nonzero) of $yx^2, x^2y \in (A, S)$, but $yx^2 - x^2y = yx^2 + xyx - x^2y - xyx = (xy + yx)x - x(xy + yx) = z^2x - xz^2 = 0$, since $xz = zx$ in A . Because of $A = \mathbb{K}\langle x, y, z \rangle / \langle z^2 - xy - yx, zx - xz, zy - yz \rangle \cong \sigma(\mathbb{K}[z]\langle x, y \rangle)$ is a graded skew PBW extension of the PBW algebra $\mathbb{K}[z]$, in this case the Proposition 4.1 fails. So it is important the order of the generators of the free algebra L as in the proof of the Proposition 4.1; for the graded skew PBW extension $A = \sigma(\mathbb{K}[z]\langle x, y \rangle)$ we have that $A = \mathbb{K}\langle z, x, y \rangle / \langle z^2 - xy - yx, zx - xz, zy - yz \rangle$, i.e., $z = x_1 < x = x_2 < y = x_3$. In this case we write the relations as $yx = -xy + z^2; xz = zx; yz = zy$, whereby $(3, 2), (2, 1), (3, 1) \notin S$. So, $S = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$, $S^{(p)} = \{(i_1, i_2, \dots, i_p) \mid i_1 \leq i_2 \leq \dots \leq i_p\}$ and $(A, S) = \{z^{\alpha_1} x^{\alpha_2} y^{\alpha_3} \mid \alpha_1, \alpha_2, \alpha_3 \geq 0\}$ is a PBW base for A .

Theorem 4.3 ([14], Theorem 5.3). *If B is a PBW algebra then B is a Koszul algebra.*

The proof of the previous theorem can be also found in [13], Theorem 3.1, page 84; they also exhibit an example of a Koszul algebra which is not a PBW algebra.

Corollary 4.4. *Let A be a graded skew PBW extension of a finitely presented algebra R . If R is a PBW algebra then A is Koszul algebra.*

Proof. From Proposition 4.1 and Theorem 4.3. □

Example 4.5. Let $R = \mathbb{K}[t_1, \dots, t_m]$ be the classical polynomial ring. Then from Corollary 4.4 every graded skew PBW extension of R is Koszul. Therefore, Examples 2.9 are Koszul algebras. Also, by Remark 4.2 and Corollary 4.4, we have that $A = \mathbb{K}\langle z, x, y \rangle / \langle z^2 - xy - yx, zx - xz, zy - yz \rangle$ is a Koszul algebra. Note that $A = \mathbb{K}\langle z, x, y \rangle / \langle z^2 - xy - yx, zx - xz, zy - yz \rangle = \sigma(\mathbb{K}[z]\langle x, y \rangle) = \mathbb{K}[z][x; \sigma_1, \delta_1][y; \sigma_2, \delta_2]$ is a graded iterated Ore extension, where $\sigma_1(z) = z$, $\sigma_2(x) = -x$, $\delta_1(z) = 0$ and $\delta_2(x) = z^2$. So, we also can be use the Proposition 3.1 to guarantee that A is Koszul.

Remark 4.6. (i) Some of the algebras in Example 2.9 had already been presented by other authors as Koszul algebras using other characterizations. For example, Smith in [21], Proposition 12.1, showed that the homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ is Koszul.

(ii) The converse of Corollary 4.4 is false. Indeed, the \mathbb{K} -algebra R minimal in the numbers of generators and relations for algebraically closed field \mathbb{K} with relations $x^2 + yz = 0$ and $x^2 + azy = 0$, $a \neq 0, 1$, is Koszul but R is not a PBW algebra (see [13], Example of page 84). The associated graded Ore extension $A := R[u]$ is Koszul algebra ([12], Corollary 1.3) and graded skew PBW extension.

(iii) Let R as in the part (ii) above. Note that $A = R[u] \cong \mathbb{K}\langle x, y, z, u \rangle / \langle x^2 + yz, x^2 + azy, ux - xu, uy - yu, uz - zu \rangle$, with $a \neq 0, 1$. So, $x < y < z < u$ and $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 4)\}$. Therefore $(1, 1, 2), (2, 1, 1) \in S^{(3)}$ and x^2y, yx^2 are nonzero monomials in A , but $a^{-1}yx^2 + x^2y = yzy - yzy = 0$. Then (A, S) is not a PBW basis, i.e., A is not a PBW algebra. So, if A is a graded skew PBW extension of the Koszul algebra R does not imply that A is PBW algebra.

(iv) With the above reasoning we have that not any graded skew PBW extension is a PBW algebra.

(v) We have also that not all graded skew PBW extension are Koszul. Indeed, let $R = \mathbb{K}\langle x, y \rangle / \langle y^2 - xy, y^2 \rangle$ be a homogeneous quadratic non-Koszul algebra ([6], page 10), then $R[u]$ is a non-Koszul associate graded Ore extension of R , which is also a graded skew PBW extension.

5 Lattices

A *lattice* is a discrete set Ω endowed with two idempotent (i.e., $a \cdot a = a$) commutative, and associative binary operations $\wedge, \vee : \Omega \times \Omega \rightarrow \Omega$ satisfying the following *absorption identities*: $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$. A lattice is called *distributive* if it satisfies the following distributivity identity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let W be a vector space. The set Ω_W of all its linear subspaces is a lattice with respect to the operations of sum and intersection. Given X_1, \dots, X_z subspaces of a vector space W , we may consider the *sublattice* of subspaces of W generated by X_1, \dots, X_z by the operations of intersection and summation. We will say that a *collection of subspaces* $X_1, \dots, X_z \subseteq W$ is *distributive* if it generates a distributive lattice of subspaces of W .

Proposition 5.1 ([13], Proposition 1-7.1). *Let W be a vector space and $X_1, \dots, X_z \subseteq W$ be a collection of its subspaces. Then the following conditions are equivalent:*

- (i) *the collection X_1, \dots, X_z is distributive;*

- (ii) *there exists a direct sum decomposition $W = \bigoplus_{j \in J} W_j$ of the vector space W such that each of the subspaces X_i is the sum of a set of subspaces W_j .*
- (iii) *there exists a basis $\mathcal{B} = \{w_i \mid i \in I\}$ of the vector space W such that each of the subspaces X_i is the linear span of a set of vectors w_i .*
- (iv) *there exists a basis \mathcal{B} of the vector space W such that $\mathcal{B} \cap X_i$ is a basis of the subspace X_i for each $1 \leq i \leq z$ ([2], Lemma 1.2).*

Let $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is a two-sided ideal generated by homogeneous elements and let $A_+ = \bigoplus_{p>0} A_p$. The *lattice associated* to A , $\Omega(A)$ is the lattice generated by $\{A_+^\lambda I^\mu A_+^\nu \mid \lambda, \mu, \nu \geq 0\} \subseteq \{\text{Subspaces of } \mathbb{K}\langle x_1, \dots, x_n \rangle\}$, where $I^0 = \mathbb{K}\langle x_1, \dots, x_n \rangle$, $I^1 = I$; $I^2 = \{\sum xy \mid x, y \in I\}$, \dots . The lattice generated by $\{A_+^\lambda I^\mu A_+^\nu \mid \lambda, \mu, \nu \geq 0, \lambda + \mu + \nu = j\}$ is denoted by $\Omega_j(A)$. Backelin in [2] shows that A is Koszul if and only if A is quadratic and $\Omega(A)$ is distributive and that $\Omega(A)$ is distributive if and only if for all $j \geq 2$, $\Omega_j(A)$ is distributive. So, A is Koszul if and only if A is quadratic and $\Omega_j(A)$ is distributive, for all j . As a consequence of this, Polishchuk and Positselski in [13] show the following criteria for Koszulness.

Lemma 5.2 ([13], Theorem 2-4.1). *A homogeneous quadratic algebra $A = L/\langle P \rangle$ ($L = \mathbb{K}\langle x_1, \dots, x_n \rangle$) is Koszul if and only if for all $k \geq 0$, the collection of subspaces*

$$X_i := L_{i-1} P L_{k-i-1} \subset L_k, \quad i = 1, \dots, k-1 \quad (5.1)$$

is distributive.

Let $R = \mathbb{K}\langle t_1, \dots, t_n \rangle / I$ be a homogeneous quadratic algebra. Note that $\Omega(R)$ only depend on the subalgebra of R generated by those of the generators of R which “appear” in a set of minimal relations for R .

Lemma 5.3 ([2], Lemma 2.3). *Let $R = \mathbb{K}\langle t_1, \dots, t_n \rangle / I$ be a quadratic algebra and let*

$$A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle / \langle I \rangle,$$

where $\langle I \rangle$ is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle$ generated by I . Then $\Omega(R)$ is distributive if and only if $\Omega(A)$ is distributive.

Lemma 5.4. *A quadratic algebra $R = \mathbb{K}\langle t_1, \dots, t_m \rangle / I$ is Koszul if and only if*

$$A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle / \langle I \rangle$$

is Koszul, where $\langle I \rangle$ is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle$ generated by I .

Proof. Note that R is quadratic if and only if A is quadratic. Also, by Lemma 5.3, $\Omega(R)$ is distributive if and only if $\Omega(A)$ is distributive. Therefore, by Lemma 5.2, R is Koszul if and only if A is Koszul. \square

Related to Proposition 3.1 we have the following theorem.

Theorem 5.5. *If $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ is a graded skew PBW extension of a finitely presented Koszul algebra R , then A is Koszul.*

Proof. Let R be a finitely presented algebra; by Proposition 2.1,

$$R = \mathbb{K}\langle t_1, \dots, t_m \rangle / \langle P \rangle \quad (5.2)$$

where P is the \mathbb{K} -space generated by homogeneous polynomials

$$r_1, \dots, r_s \in L^R := \mathbb{K}\langle t_1, \dots, t_m \rangle. \quad (5.3)$$

Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a graded skew PBW extension. Then by Remark 2.10, A is a finitely presented algebra. So, by Proposition 2.1,

$$A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle / \langle W \rangle, \quad (5.4)$$

where W is the \mathbb{K} -space generated by the polynomials

$$\begin{aligned} r_1, \dots, r_s, \quad x_j t_k - \sigma_j(t_k) x_i - \delta_j(t_k), \quad x_j x_i - c_{i,j} x_i x_j - (r_{0,j,i} + r_{1,j,i} x_1 + \dots + r_{n,j,i} x_n) \\ \in L := \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle, \end{aligned} \quad (5.5)$$

with $1 \leq i, j \leq n, 1 \leq k \leq m$.

Since R is a Koszul algebra then:

- (i) By Lemma 5.2 we have that R is homogeneous quadratic algebra, and by Remark 2.10, A is homogeneous quadratic algebra.
- (ii) By Lemma 5.4, we have that $A_P := \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle / \langle P \rangle_X$ is Koszul, where $\langle P \rangle_X$ is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n \rangle$ generated by the polynomials as in 5.3. So, by Lemma 5.2, we have that for all $k \geq 0$, the collection of subspaces

$$X_i^P := L_{i-1} P L_{k-i-1} \subseteq L_k, \quad i = 1, \dots, k-1 \quad (5.6)$$

is distributive. Therefore, by Proposition 5.1, there exist a basis \mathcal{B}_k of the space L_k such that $\mathcal{B}_k \cap X_i^P$ is a basis of X_i^P for each $1 \leq i \leq k-1$. Let $X_i := L_{i-1} W L_{k-i-1} \subseteq L_k$, $i = 1, \dots, k-1$, where W is the space generated by the polynomials as in (5.5).

Let $Y := (X_i \setminus X_i^P) \cap \mathcal{B}_k$. Since X_i^P is a subspace of X_i we claim that $\bar{Y} := \{y + X_i^P \mid y \in Y\} = \{\bar{y} \in X_i/X_i^P \mid y \in Y\}$ is a basis of X_i/X_i^P . Indeed: if $0 \neq \bar{x} \in X_i/X_i^P$, then $\bar{x} = x + X_i^P$, with $x \in X_i \setminus X_i^P$. Note that $x = k_1 b_1 + \dots + k_\rho b_\rho$, where b_1, \dots, b_ρ are different nonzero elements in \mathcal{B}_k and $k_1, \dots, k_\rho \in \mathbb{K}$. Then $\bar{x} = \overline{k_1 b_1 + \dots + k_\rho b_\rho} = k_1 \bar{b}_1 + \dots + k_\rho \bar{b}_\rho$. If $b_\nu \in X_i^P$ for some $1 \leq \nu \leq \rho$ then $\bar{b}_\nu = 0$. So $\bar{x} = s_1 \bar{v}_1 + \dots + s_\mu \bar{v}_\mu$ with $s_1, \dots, s_\mu \in \mathbb{K}$ and $v_1, \dots, v_\mu \in Y$. Now suppose that $k_1 \bar{y}_1 + \dots + k_v \bar{y}_v = \bar{0}$ with $k_1, \dots, k_v \in \mathbb{K}$ and $0 \neq \bar{y}_1, \dots, 0 \neq \bar{y}_v \in \bar{Y}$. Then $y_1, \dots, y_v \notin X_i^P$, $k_1 y_1 + \dots + k_v y_v = 0$ and so $k_1 y_1 + \dots + k_v y_v \in X_i^P$. As $X_i^P \cap \mathcal{B}_k$ is a basis of X_i^P then there are different nonzero elements $w_{v+1}, \dots, w_{v+\mu} \in X_i^P \cap \mathcal{B}_k$ such that $k_1 y_1 + \dots + k_v y_v = k_{v+1} w_{v+1} + \dots + k_{v+\mu} w_{v+\mu}$, with $k_{v+1}, \dots, k_{v+\mu} \in \mathbb{K}$. As $y_1, \dots, y_v \notin X_i^P$ then $y_1, \dots, y_v, w_{v+1}, \dots, w_{v+\mu}$ are nonzero different elements in \mathcal{B}_k such that

$k_1 y_1 + \cdots + k_v y_v + (-k_{v+1}) w_{v+1} + \cdots + (-k_{v+\mu}) w_{v+\mu} = 0$. Then $k_1 = \cdots = k_v = k_{v+1} = \cdots = k_{v+\mu} = 0$.

Therefore, by Theorem 3.33 in [22], we have that $(\mathcal{B}_k \cap X_i^P) \cup Y = \mathcal{B}_k \cap X_i$ is a basis of X_i . So, by Proposition 5.1 the collection of subspaces X_1, \dots, X_{k-1} is distributive for each $k \geq 0$. Whence, by Lemma 5.2 we have that A is Koszul. □

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